

On Bidding with Securities: Risk Aversion and Positive Dependence

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Abstract

DeMarzo et al. (2005) considers auctions in which bids are selected from a completely ordered family of securities whose ultimate values are tied to the resource being auctioned. The paper defines a notion of relative *steepness* of families of securities and shows that a steeper family provides a higher expected revenue to the seller. Two key assumptions are: (i) the buyers are risk-neutral; (ii) the random variables through which values and signals of the buyers are realized are affiliated. This paper studies the role of these assumptions and the consequences of relaxing them in the case of the second price auction. We show that the revenue ranking of families of securities of DeMarzo et al. (2005) holds for risk averse buyers. However, this ranking is not preserved if affiliation is relaxed to a less restrictive form of positive dependence among values and signals, namely, first order stochastic dominance. We then define the relative *strong steepness* of families of securities and show that a strongly steeper family generates a higher expected revenue for the seller in the case of this more general form of positive dependence.

1 Introduction

Consider auctioning an asset that is a resource to be developed for profit by the winning buyer. It is common in such auctions to require bids in the form of securities whose values to the seller are tied to the eventual realized value of the asset. As an alternative to simply soliciting cash bids for the asset, for instance, a seller may require buyers to compete in terms of the equity share that the seller retains of the asset's profits. Other common securities used in bidding include debt and call options. DeMarzo et al. (2005) develops a general theory of bidding with securities in the first price and the second price auctions. Bids are selected from a completely ordered family of securities and the paper focuses on the importance of the design of the family of securities to the seller's expected revenue. The paper defines a partial ordering of families based on the notion of *steepness* (to be made precise in Section 3) and shows that the steeper family of securities provides higher expected revenue to the seller. Two assumptions are made to prove this result: (i) buyers are *risk-neutral*; (ii) the random variables through which values and signals of the buyers are realized are *affiliated*.

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Risk neutrality is a severe restriction for a financial model. Affiliation is an extremely restrictive form of positive dependence.¹

Our objective in this paper is to explore in the case of the second price auction the dependence of the revenue ranking of families of securities upon these two assumptions.² We work with a symmetric interdependent values model on the lines of Milgrom and Weber (1982) and risk averse buyers. We consider two additional forms of positive dependence, namely, the *monotone likelihood ratio* (henceforth, MLR) property, which is strictly weaker than affiliation;³ and *first order stochastic dominance* (henceforth, FOSD), which is strictly weaker than the MLR property. Each of these three positive dependence conditions has been extensively used in both auction theory and information economics.

Our main results are the following:

- (i) A steeper family of securities provides higher expected revenue to the seller even with risk averse buyers and assuming that the values are positively dependent on signals in the MLR sense. We in this sense extend the result of DeMarzo et al. (2005) to the case of risk aversion.⁴
- (ii) We show by an example that if the notion of positive dependence among values and signals of buyers is relaxed further from MLR to FOSD, then even for risk neutral buyers the revenue ranking of families of securities of DeMarzo et al. (2005) no longer holds.
- (iii) We present an appropriate modification of the revenue ranking of families of securities that holds for risk averse or risk neutral buyers, and for FOSD as the notion of positive dependence among values and signals. This ranking is based upon a condition that we call *strong steepness* – the strongly steeper family of securities provides higher expected revenue to the seller.

This paper is organized as follows. Section 2 outlines our model, notation, and definitions. Section 3 extends the revenue ranking of families of securities of DeMarzo et al. (2005) to risk averse buyers. Section 4 shows that this ranking is not preserved under a more general form of positive dependence, i.e., FOSD. A modification of the revenue ranking of families of securities based on strong steepness is then presented. We conclude in Section 5.

¹de Castro (2010) shows that the set of affiliated probability density functions for two random variables is the complement of an open and dense set in the space of continuous probability density functions under an appropriate topology and has zero measure under an appropriate measure.

²While our paper is restricted to the case of the second price auction, DeMarzo et al. (2005) also ranks families of securities in the case of the first price auction. An additional restriction on the set of securities and the dependence of values and signals beyond affiliation is required in this analysis (i.e., the *log-supermodularity* of each buyer's expected profit, which is Assumption C in the paper). Our interest in this paper is in exploring the effect of relaxing the assumption of affiliation and not restricting it further. We have not been able to carry out the analysis for the first price auction at this level of generality.

³DeMarzo et al. (2005) assumes the MLR property for the case of independent private values and affiliation for the case of interdependent values. For independent private values, the MLR property and affiliation are equivalent but not for interdependent values. Section 2 provides further details.

⁴It is also worth emphasizing that our proofs are more straightforward than those in DeMarzo et al. (2005) and do not require its strong regularity assumption (Assumption B) on the probability density of return conditioned on a buyers signal. We accomplish this mainly by exploiting the properties of concave functions. Furthermore, since affiliation implies the MLR property, a byproduct of our result is an extension of the result of DeMarzo et al. (2005) to a richer informational environment.

2 Model, Notation, and Assumptions

Consider N buyers competing for a resource that a seller wants to sell. Each buyer has a certain value for the resource that is unknown to him; however, each buyer has some information (henceforth, *signal*) about the value of the resource. The signal of a buyer is known only to him, but it may be informative to other buyers in the sense that it may improve their respective estimates of the value of the resource.

We model this by assuming that the value of the resource to a buyer n , denoted by x_n , is a realization of a nonnegative random variable X_n , unknown to him. This is the profit to buyer n from developing the resource in the absence of any payments to the seller, but after taking into account the variable costs. A buyer n privately observes a signal y_n through a realization of a random variable Y_n that is correlated with (X_1, X_2, \dots, X_N) . A winning buyer needs to invest a fixed amount $I > 0$, which is the same for each buyer, to develop the resource. As in DeMarzo et al. (2005), we assume that the realization of X_n is observed ex-post by the seller and buyer n if buyer n wins and subsequently uses the resource. The joint cumulative distribution function (CDF) of the random variables X_n 's and Y_n 's is common knowledge.

Let $\mathbf{x} \triangleq (x_1, x_2, \dots, x_N)$ denote a vector of values; denote the random vector (X_1, X_2, \dots, X_N) by \mathbf{X} . A vector of signals \mathbf{y} and the random vector \mathbf{Y} are defined similarly. We use the standard game theoretic notation of $\mathbf{x}_{-n} \triangleq (x_1, \dots, x_{n-1}, x_{n+1}, \dots, x_N)$. Similar interpretations are used for \mathbf{X}_{-n} , \mathbf{y}_{-n} , and \mathbf{Y}_{-n} . Define the random variable Z_1 as the largest among Y_2, Y_3, \dots, Y_N ; i.e., $Z_1 \triangleq \max(Y_2, Y_3, \dots, Y_N)$. Denote a realization of Z_1 by z_1 .

Let $F_{\mathbf{X}, \mathbf{Y}}(\mathbf{x}, \mathbf{y})$ denote the joint CDF of (\mathbf{X}, \mathbf{Y}) . It is assumed to have the following symmetry property:

Assumption 1. The joint CDF of $(X_n, Y_n, \mathbf{Y}_{-n})$, denoted by $F_{X_n, Y_n, \mathbf{Y}_{-n}}(x_n, y_n, \mathbf{y}_{-n})$, is identical for each n and is symmetric in its last $N - 1$ components (i.e., in \mathbf{y}_{-n}).

Assumption 1 allows for a special dependence between the value of the resource to a buyer and his own signal, while the identities of other buyers are irrelevant to him. The model reduces to the independent private values model if (X_n, Y_n) is independent of $(\mathbf{X}_{-n}, \mathbf{Y}_{-n})$ for all n , to the pure common value model if $X_1 = X_2 = \dots = X_N$, and includes a continuum of interdependent value models between these two extremes. Because of Assumption 1, the subsequent assumptions and analysis are given from buyer 1's viewpoint.

For each n , the set of possible values that X_n can take is assumed to be an interval $[\underline{x}, \bar{x}]$ and the set of possible values that Y_n can take is assumed to be an interval $[\underline{y}, \bar{y}]$. Assume that the joint probability density function (pdf) of the random vector \mathbf{Y} , denoted by $f_{\mathbf{Y}}(\mathbf{y})$, exists and is positive for all $\mathbf{y} \in [\underline{y}, \bar{y}]^N$. By Assumption 1, $f_{\mathbf{Y}}(\mathbf{y})$ is symmetric in its N arguments.

Let larger numerical values of the signals correspond to more favorable estimates of the value of the resource. There are several ways of formalizing the idea that larger signals correspond to larger values of the resource. The following three ways are commonly considered:

Definition 1 (Affiliation). Assume that the joint pdf of (X_1, \mathbf{Y}) , denoted by $f_{X_1, \mathbf{Y}}(x_1, \mathbf{y})$, exists and is positive everywhere on $[\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}]^N$. The random variables (X_1, \mathbf{Y}) are *affiliated* if

$$f_{X_1, \mathbf{Y}}((x_1, \mathbf{y}) \vee (\hat{x}_1, \hat{\mathbf{y}})) f_{X_1, \mathbf{Y}}((x_1, \mathbf{y}) \wedge (\hat{x}_1, \hat{\mathbf{y}})) \geq f_{X_1, \mathbf{Y}}(x_1, \mathbf{y}) f_{X_1, \mathbf{Y}}(\hat{x}_1, \hat{\mathbf{y}}),$$

for any (x_1, \mathbf{y}) and $(\hat{x}_1, \hat{\mathbf{y}})$ in the support of (X_1, \mathbf{Y}) . Here “ \vee ” denotes coordinatewise maximum and “ \wedge ” denotes coordinatewise minimum.

Definition 2 (PD-MLR). Assume that for any y_1 and z_1 , the pdf of X_1 conditioned on $Y_1 = y_1$ and $Z_1 = z_1$, denoted by $f_{X_1|Y_1=y_1, Z_1=z_1}(x_1)$, exists and is positive everywhere on $[\underline{x}, \bar{x}]$. The random variable X_1 is *positively dependent (PD) on the random variables (Y_1, Z_1) in the maximum likelihood ratio (MLR) sense* if $f_{X_1|Y_1=y_1, Z_1=z_1}(x_1)/f_{X_1|Y_1=\hat{y}_1, Z_1=\hat{z}_1}(x_1)$ is nondecreasing in x_1 for any $y_1 \geq \hat{y}_1$ and $z_1 \geq \hat{z}_1$.

Definition 3 (PD-FOSD). The random variable X_1 is *positively dependent (PD) on the random variables (Y_1, Z_1) in the first order stochastic dominance (FOSD) sense* if for any x_1 , $1 - F_{X_1|Y_1=y_1, Z_1=z_1}(x_1)$ is nondecreasing in y_1 and z_1 , where $F_{X_1|Y_1=y_1, Z_1=z_1}(x_1)$ is the CDF of X_1 conditioned on $Y_1 = y_1$ and $Z_1 = z_1$.

Under Assumption 1, the following relationship between affiliation, PD-MLR, and PD-FOSD is well known in the existing literature:⁵

Lemma 1. *Affiliation implies PD-MLR and PD-MLR implies PD-FOSD.*

Lemma 2. *PD-FOSD is equivalent to $\mathbb{E}[h(X_1)|Y_1 = y_1, Z_1 = z_1]$ being nondecreasing in y_1 and z_1 for any nondecreasing function $h : \mathbb{R} \mapsto \mathbb{R}$ for which the expectation exists.*

Our focus is on comparing PD-MLR and PD-FOSD. Lemma 1 implies that results obtained by assuming PD-FOSD hold if PD-MLR is assumed instead, and results obtained by assuming PD-MLR hold if affiliation is assumed instead.⁶ It is common in auction theory to justify the assumption of either affiliation or the MLR property by citing the plausibility that $\mathbb{E}[h(X_1)|Y_1 = y_1, Z_1 = z_1]$ is nondecreasing in y_1 and z_1 for any nondecreasing function h . However, the relationship in Lemma 1 does not go in the reverse direction; PD-MLR is strictly stronger than PD-FOSD. Section 4 provides further details.

The buyers are assumed to be risk averse or risk neutral. Each buyer has the same von Neumann-Morgenstern utility of money, denoted by $u : \mathbb{R} \rightarrow \mathbb{R}$, which is concave (possibly linear), increasing, and normalized so that $u(0) = 0$. Henceforth, the term *risk averse* includes risk neutral behavior. The seller is risk neutral. Conditioned on any y_1 and z_1 , the expected utility of the resource to buyer 1 without any payments is assumed to be positive; i.e., $\mathbb{E}[u(X_1 - I)|Y_1 = y_1, Z_1 = z_1] > 0$. Thus, the buyers who compete for the resource expect to make a positive profit from utilizing it.

As in DeMarzo et al. (2005), buyers bid with securities from some ordered family. Let $\Phi \triangleq \{\phi(\cdot, b) \mid b \in [\underline{b}, \bar{b}]\}$ be a family of securities parametrized by b . A bid b of buyer 1 denotes his willingness to pay an amount $\phi(x_1, b)$ to the seller if $X_1 = x_1$. The interval $[\underline{b}, \bar{b}]$ can be normalized to any arbitrary closed interval, independently of ϕ , by translation and rescaling of the parameter b in $\phi(\cdot, b)$. The family Φ is assumed to satisfy the following conditions:

Assumption 2. For any b , $\phi(x_1, b)$ is continuous and nondecreasing in x_1 , and $x_1 - \phi(x_1, b)$ is nondecreasing and nonconstant in x_1 .

Assumption 2 says that the payment made to the seller is nondecreasing in the realized value of the resource. Moreover, the profit of the winning buyer is also nondecreasing in the realized value of the resource.

⁵Assumption 1 is used in showing that if the random variables (X_1, \mathbf{Y}) are affiliated then so are the random variables (X_1, Y_1, Z_1) ; see Milgrom and Weber (1982). Lemma 1 then follows from the known relationship between affiliation, MLR, and FOSD; see, e.g., Shaked and Shanthikumar (2006). We use the prefix “PD-” with MLR or FOSD to indicate that positive dependence is assumed here in reference to X_1 and (Y_1, Z_1) .

⁶DeMarzo et al. (2005) assumes a strict MLR property for the independent private values case and strict affiliation for the interdependent values case.

Assumption 3. For any y_1 and z_1 ,

- (i) $\mathbb{E}[u(X_1 - I - \phi(X_1, b)) | Y_1 = y_1, Z_1 = z_1]$ is continuous and decreasing in b , nonnegative for $b = \underline{b}$, and nonpositive for $b = \bar{b}$.
- (ii) $\mathbb{E}[\phi(X_1, b) | Y_1 = y_1, Z_1 = z_1]$ is continuous and increasing in b .

Assumption 3 says the family of securities is completely ordered, independently of the realized signal vector. The seller prefers higher security bids while the buyers prefer lower security bids. Assumption 3 is satisfied if, e.g., for any x_1 , $\phi(x_1, b)$ is increasing in b . The seller uses the second price auction where the highest bidder wins and pays according to the security bid of the second highest bidder. The boundary conditions in Assumption 3(i) guarantee the existence of a pure strategy equilibrium for the second price auction. Some common families of securities that satisfy Assumptions 2 and 3 are: *pure cash bid* $\phi(x_1, b) = b$, $b \in [0, \bar{x}]$; *debt* $\phi(x_1, b) = \min(x_1, b)$, $b \in [0, \bar{x}]$; *equity* $\phi(x_1, b) = bx_1$, $b \in [0, 1]$; and *call option* $\phi(x_1, b) = [x_1 - \bar{x} + b]^+$, $b \in [0, \bar{x}]$. These families of securities are shown in Figure 1.

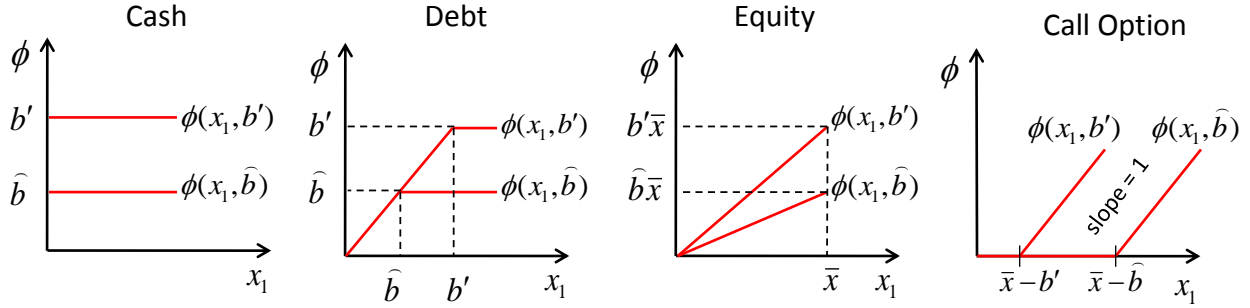


Figure 1: Plot of some families of securities (pure cash bid, debt, equity, and call option) for $b' > \hat{b}$.

Assumptions 1-3 are in place for rest of this paper. For a comparison between two different families of securities, we use $\Psi \triangleq \{\psi(\cdot, b) \mid b \in [\underline{b}, \bar{b}]\}$ to denote a family of securities different from Φ . All expectations and conditional expectations of interest are assumed to exist and be finite. In any further usage, x_1 , y_n , and z_1 are always in the support of random variables X_1 , Y_n , and Z_1 , respectively.

3 Risk Aversion

This section extends the result of DeMarzo et al. (2005) on revenue ranking of families of securities to risk averse buyers. In a second price auction, a buyer n decides how much to bid solely based on his signal y_n . We look for a symmetric equilibrium. We start by defining a function $s(y_1, z_1; \Phi)$ that will be used to characterize the bidding strategies of the buyers:

$$s(y_1, z_1; \Phi) \triangleq \{b : \mathbb{E}[u(X_1 - I - \phi(X_1, b)) | Y_1 = y_1, Z_1 = z_1] = 0\}. \quad (1)$$

The function s is well defined because of Assumption 3. If bids are restricted to the family of securities Φ , then $s(y_1, z_1; \Phi)$ is the highest bid that buyer 1 is willing to submit conditioned on

his signal being y_1 and the highest signal of other buyers being z_1 . Notice that the bid $s(y_1, z_1; \Phi)$ corresponds to buyer 1's willingness to pay an amount $\phi(x_1, s(y_1, z_1; \Phi))$ to the seller if $X_1 = x_1$. The next lemma characterizes some important properties of the function s .

Lemma 3. *Assuming PD-FOSD, the function $s(y_1, z_1; \Phi)$ is nondecreasing in y_1 and z_1 .*

Proof. Since $x_1 - I - \phi(x_1, b)$ is nondecreasing in x_1 by Assumption 2 and u is an increasing function, the claim follows immediately from Lemma 2. \square

To simplify the analysis in the rest of this paper, we reinforce Lemma 3 with the following additional assumption:

Assumption 4. The family of securities and the informational environments are such that the function $s(y_1, z_1; \Phi)$ is *increasing* in y_1 .

Assumption 4 greatly simplifies the analysis by ruling out the possibility of ties among bids, which is therefore an event that we can ignore in the remainder of the paper. Assumption 4 is satisfied in most cases of interest, e.g., if for any $x_1 \in (\underline{x}, \bar{x})$, $1 - F_{X_1|Y_1=y_1, Z_1=z_1}(x_1)$ is increasing in y_1 . The results of this paper hold without Assumption 4, though the analysis is more complicated.

The next lemma characterizes an equilibrium bidding strategy for the second price auction with bids restricted to the family Φ of securities. The construction of the bidding strategy is on the lines of Milgrom and Weber (1982).

Lemma 4. *Let the strategies $\beta_1, \beta_2, \dots, \beta_N$ of the buyers be identical and defined by $\beta_n(y_n) \triangleq s(y_n, y_n; \Phi)$ for all n . Assuming PD-FOSD, the strategy vector $(\beta_1, \beta_2, \dots, \beta_N)$ is a symmetric Bayes-Nash equilibrium of the second price auction with bids restricted to the family Φ .*

Proof. Assume that each buyer n except buyer 1 uses the strategy $\beta_n(y_n) = s(y_n, y_n; \Phi)$. We will show that the best response for buyer 1 is to use the strategy $\beta_1(y_1) = s(y_1, y_1; \Phi)$.

Given y_1 , let buyer 1 bid b . Buyer 1 wins if $b \geq \max\{s(y_n, y_n; \Phi) : 2 \leq n \leq N\}$. From Lemma 3, $\max\{s(y_n, y_n; \Phi) : 2 \leq n \leq N\} = s(z_1, z_1; \Phi)$, where $z_1 = \max\{y_2, y_3, \dots, y_N\}$. Thus, the expected utility of buyer 1 is given by:

$$\begin{aligned} \mathbb{E} [u(X_1 - I - \phi(X_1, s(Z_1, Z_1; \Phi))) \mathbf{1}_{\{b \geq s(Z_1, Z_1; \Phi)\}} | Y_1 = y_1] \\ = \mathbb{E} [\mathbb{E} [u(X_1 - I - \phi(X_1, s(Z_1, Z_1; \Phi))) | Y_1 = y_1, Z_1] \mathbf{1}_{\{b \geq s(Z_1, Z_1; \Phi)\}} | Y_1 = y_1]. \end{aligned}$$

From (1), $\mathbb{E} [u(X_1 - I - \phi(X_1, s(y_1, Z_1; \Phi))) | Y_1 = y_1, Z_1] = 0$, and from Assumption 4, $s(y_1, z_1; \Phi)$ is increasing in y_1 . Hence, $\mathbb{E} [u(X_1 - I - \phi(X_1, s(Z_1, Z_1; \Phi))) | Y_1 = y_1, Z_1]$ is positive for $Z_1 < y_1$ and negative for $Z_1 > y_1$. The expected utility of buyer 1 is uniquely maximized by setting $b = s(y_1, y_1; \Phi)$. \square

Because of symmetry, the seller's expected revenue equals the expected payment made by buyer 1 conditioned on him winning. In the symmetric equilibrium given by Lemma 4, the bid of buyer 1 is the highest if and only if his signal is the highest among all the buyers (i.e., $y_1 > z_1$). If buyer 1 wins, his payment is determined by the second highest security bid (i.e., $s(z_1, z_1; \Phi)$). Thus, the seller's expected revenue from the second price auction with bids restricted to the family Φ is $\mathbb{E} [\phi(X_1, s(Z_1, Z_1; \Phi)) | Y_1 > Z_1]$.

We next reformulate the definition of steepness from DeMarzo et al. (2005) using the concept of quasi-monotonicity, as defined below:

Definition 4 (Quasi-monotone function). A function $g(w)$ is *quasi-monotone* if for any w and \hat{w} such that $w > \hat{w}$, if $g(\hat{w}) > 0$ then $g(w) \geq 0$. A quasi-monotone function therefore crosses zero at most once and from below.

Definition 5 (Steepness). A family of securities Φ is *steeper* than another family of securities Ψ if for any $b', \hat{b} \in [\underline{b}, \bar{b}]$, $\phi(w, b') - \psi(w, \hat{b})$ is quasi-monotone in w .

Notice that call option is steeper than equity, which is steeper than debt, which is steeper than cash (see Figure 1).⁷

Proposition 1 below gives a sufficient condition under which two different families of securities can be ranked according to the revenue they generate. The proof is in Appendix A.

Proposition 1. *Let Φ and Ψ be two families of securities such that Φ is steeper than Ψ . Assuming PD-MLR, the second price auction with bids restricted to the family Φ generates at least as much expected revenue for the seller as the second price auction with bids restricted to the family Ψ .*

A careful review of the proof of Proposition 1 shows that we in fact prove the stronger result that the expected revenue of the seller conditioned on the winning buyer's signal and the second highest signal is at least as large in the case of the steeper family of securities Φ as with the family Ψ . The revenue from the steeper family thus weakly dominates in this ex-post sense, which implies that it is weakly better for the seller ex-ante as stated in the proposition. This remark also applies to Proposition 3 in Section 4.

The following is an immediate consequence of Proposition 1:

Corollary 1. Assuming PD-MLR, the expected revenue from the following families of securities can be ranked as: pure cash bid \leq debt \leq equity \leq call option.

The revenue ranking of families of securities of DeMarzo et al. (2005) is essentially Proposition 1 and Corollary 1 for risk neutral buyers.

4 Positive Dependence

This section addresses the role of the positive dependence assumption in the ranking of families of securities. An example is first discussed that shows that the ranking of Proposition 1 does not hold if PD-MLR is relaxed to PD-FOSD.⁸ The pairwise ranking of the three families of securities – debt, equity, and call options – is completely reversed in this example in comparison to the ranking according to Corollary 1. If PD-MLR is relaxed to PD-FOSD, the steepness condition must be strengthened in order to rank two families of securities. This is accomplished by using the notion of *strong steepness* that we define below.

⁷Quasi-monotonicity is not transitive and hence steepness is not transitive. Proposition 1 provides pairwise revenue ranking for any two families of securities that are ordered under the steepness criteria. This revenue ranking, however, is transitive.

⁸Interestingly, the example assumes independent private values among the buyers; it does not rely upon interdependence of values and the problems of inference that it creates, which is commonly the source of problems in models of trading.

Example 1. Consider two risk neutral buyers (i.e., $u(w) = w$) with independent private values. Buyer n 's signal Y_n is uniformly distributed in the interval $[0, 1]$. Conditioned on $Y_n = y_n$, the random variable X_n , denoting the value of buyer n , has the following conditional pdf:

$$f_{X_n|Y_n=y_n}(x_n) = \begin{cases} 1 - y_n + 6x_n y_n & \text{if } 0 \leq x_n \leq \frac{1}{3}, \\ 1 & \text{if } \frac{1}{3} < x_n \leq 1. \end{cases} \quad (2)$$

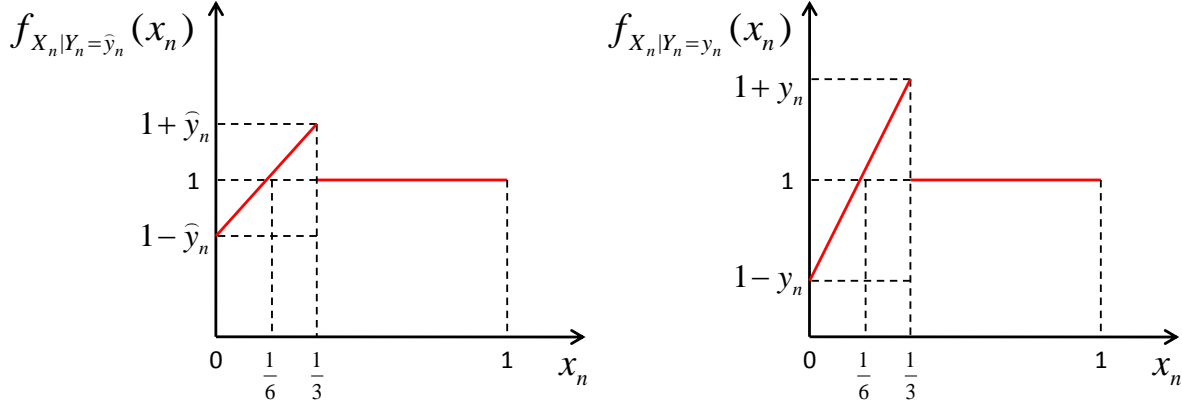


Figure 2: pdf of X_n conditioned on $Y_n = \hat{y}_n$ and conditioned on $Y_n = y_n$ for $y_n > \hat{y}_n$.

Figure 2 shows the plot of $f_{X_n|Y_n=y_n}(x_n)$. The pair (X_n, Y_n) are i.i.d. across the buyers. Since there are only two buyers, $Z_1 = Y_2$ and $F_{X_1|Y_1=y_1, Z_1=z_1}(x_1) = F_{X_1|Y_1=y_1}(x_1)$. The CDF $F_{X_n|Y_n=y_n}(x_n)$ is given by:

$$1 - F_{X_n|Y_n=y_n}(x_n) = \begin{cases} 1 - x_n + y_n(x_n - 3x_n^2) & \text{if } 0 \leq x_n \leq \frac{1}{3}, \\ 1 - x_n & \text{if } \frac{1}{3} < x_n \leq 1. \end{cases} \quad (3)$$

Since $x_n - 3x_n^2 > 0$ for $x_n \in [0, 1/3]$, $1 - F_{X_n|Y_n=y_n}(x_n)$ is increasing in y_n for $x_n \in [0, 1/3]$ and is constant in y_n for $x_n \in [1/3, 1]$. Thus, X_n is positively dependent on Y_n in the FOSD sense and PD-FOSD is satisfied (in this example, X_1 is independent of $Z_1 = Y_2$). However, for $y_n > \hat{y}_n$, $f_{X_n|Y_n=y_n}(x_n)/f_{X_n|Y_n=\hat{y}_n}(x_n)$ fails to be nondecreasing in x_n ; the ratio is strictly greater than one for $x_n \in (1/6, 1/3]$ and is equal to one for $x_n \in [1/3, 1]$. Thus, PD-MLR is not satisfied.

Example 1 highlights the distinction between PD-MLR and PD-FOSD in the following sense. If the random variable X_1 is positively dependent on the random variable Y_1 in the MLR sense (i.e., $f_{X_1|Y_1=y_1}(x_1)/f_{X_1|Y_1=\hat{y}_1}(x_1)$ is nondecreasing in x_1 for any $y_1 \geq \hat{y}_1$), then conditioning on a larger Y_1 shifts the probability distribution of X_1 towards the larger values of X_1 everywhere in the interval of possible values of X_1 . However, if the random variable X_1 is positively dependent on Y_1 in the FOSD sense (i.e., $1 - F_{X_1|Y_1=y_1}(x_1)$ is nondecreasing in y_1 for any x_1), then the shift of the probability distribution towards the larger values of X_1 when conditioned on a larger value of Y_1 can be *localized*; in Example 1, a larger value of Y_1 changes the probability distribution of X_1 only in the interval $[0, 1/3]$, making the values in $[0, 1/3]$ close to $1/3$ more likely than the values close to 0, while the likelihood of the values of X_1 in the interval $(1/3, 1]$ remains unchanged. Proposition 2 below uses this difference between PD-MLR and PD-FOSD to show that Example 1 violates the revenue ranking given by Corollary 1. The proof is in Appendix B.

Proposition 2. *For Example 1, there exists an interval of choices for investment I such that for any realization of the signal vector (Y_1, Y_2) , the expected revenue to the seller from the second price auction with bids restricted to debt securities is higher than the expected revenue from bids restricted to equity securities.*

Recall that Corollary 1 ranks the revenue from four families of securities in the case of PD-MLR as: pure cash bid \leq debt \leq equity \leq call option. Numerical computation for Example 1 with investment $I = 0.2$ results in the following values for the seller's expected revenue: from cash bids $= 0.3062$; from call option $= 0.3078$; from equity $= 0.3099$; and from debt $= 0.3123$.⁹ Thus, the ranking in Example 1 for $I = 0.2$ is: pure cash bid $<$ call option $<$ equity $<$ debt. Notice that (i) the cash bid is last in each ranking, and (ii) compared to Corollary 1, the relative pairwise ranking of debt, equity, and call option are reversed in this example. We show below in Corollary 2 that point (i) holds generally in the case of PD-FOSD, i.e., call option, equity and debt all produce a greater expected revenue for the seller than cash bids in this case. The inferiority of cash bids relative to these other securities thus generalizes from PD-MLR to PD-FOSD. Because the distributions that satisfy PD-MLR form a subset of those that satisfy PD-FOSD, the two rankings above show that any ranking of any pair of the three families of securities of debt, equity and call options is possible within the family of distributions that satisfy PD-FOSD. A general comparison of these three families of securities thus requires restricting the dependence of signals and values beyond PD-FOSD.

The next proposition gives a revenue ranking of families of securities that holds under PD-FOSD with risk averse buyers. This is achieved by strengthening the steepness condition.

Definition 6 (Strong steepness). A family of securities Φ is *strongly steeper* than another family of securities Ψ if for any $b', \hat{b} \in [\underline{b}, \bar{b}]$, $\phi(w, b') - \psi(w, \hat{b})$ is nondecreasing in w .

Notice that strong steepness implies steepness. Furthermore, debt, equity, and call option are all strongly steeper than pure cash bid.

Proposition 3. *Let Φ and Ψ be two families of securities such that Φ is strongly steeper than Ψ . Assuming PD-FOSD, the second price auction with bids restricted to the family Φ generates at least as much expected revenue for the seller as the second price auction with bids restricted to the family Ψ .*

The proof of Proposition 3 is in Appendix C. The following is an immediate consequence of Proposition 3:

Corollary 2. Assuming PD-FOSD, the expected revenue from debt, equity, or call option are more than the expected revenue from pure cash bids.

It is instructive to compare the revenue ranking of Proposition 3 to the ranking in DeMarzo et al. (2005). Recall Example 1. As noted above, PD-MLR shifts the distribution of X_1 across its support as y_1 increases while PD-FOSD may only shift this distribution locally. Steepness is fundamentally a local condition that restricts how a security from one family crosses a security from another family (i.e., it crosses at most once and from below). PD-MLR is a global notion

⁹Each of these values is obtained through a straightforward numerical integration of the expected revenue $\mathbb{E}[\phi(X_1, s(Z_1, Z_1; \phi)) | Y_1 > Z_1]$ for each family ϕ of securities. We turn to computation because of the complexity of formally calculating this value for each of the four families of securities under consideration.

of positive dependence that allows this local comparison of two families to determine a ranking based upon the seller's expected revenue. In moving from PD-MLR to PD-FOSD, however, this ranking no longer holds. Steepness is replaced in Proposition 3 by strong steepness that compares two families of securities across the entire support of X_1 . DeMarzo et al. (2005) thus apply a local condition on families of securities together with a global condition on positive dependence in order to rank families of securities in terms of expected revenue. When the global condition on positive dependence PD-MLR is weakened to the condition PD-FOSD that may only bind locally, we must strengthen the comparison of the securities to a global condition that holds across the support of X_1 in order to be able to rank the families.

We conclude with intuition on why a strongly steeper family of securities generates a higher expected revenue for the seller for the case of risk neutral buyers. Let Φ and Ψ denote two families of securities such that Φ is strongly steeper than Ψ . Consider $y_1 > z_1$, i.e., buyer 1 wins regardless of whether bids are from the family Φ or Ψ . Buyer 1 in each case pays the bid of the buyer who observed signal z_1 . His ex-post payment is equal to $\phi(x_1, s(z_1, z_1; \Phi))$ if bids are from the family Φ and the value X_1 of the resource is equal to x_1 , and the corresponding payment in the case of family Ψ is $\psi(x_1, s(z_1, z_1; \Psi))$. In our symmetric model with risk neutral buyers, $s(z_1, z_1; \Phi)$ and $s(z_1, z_1; \Psi)$ are bids that make buyer 1 indifferent to winning conditioned on $Y_1 = Z_1 = z_1$:

$$\begin{aligned} \mathbb{E}[\phi(X_1, s(z_1, z_1; \Phi)) | Y_1 = z_1, Z_1 = z_1] &= \mathbb{E}[X_1 - I | Y_1 = z_1, Z_1 = z_1] \\ &= \mathbb{E}[\psi(X_1, s(z_1, z_1; \Psi)) | Y_1 = z_1, Z_1 = z_1]. \end{aligned} \quad (4)$$

The seller would thus expect to receive the same revenue from the families Φ and Ψ if the highest and the second highest signals are the same, i.e., $Y_1 = Z_1 = z_1$. Buyer 1 wins the auction, however, when his signal $Y_1 = y_1$ is greater than z_1 . His expected payment to the seller is therefore calculated conditioned on $Y_1 = y_1 > z_1$. Intuitively, PD-FOSD means that a larger realized signal $Y_1 = y_1$ shifts the distribution of the return X_1 from the resource towards its larger values. This shift increases the expected payment to the seller from the strongly steeper family of securities Φ more than that from Ψ because the ex-post payment to the seller increases more rapidly as a function of x_1 in the case of steeper security. Hence, compared to $Y_1 = z_1$ for which we have the equality (4), for $Y_1 = y_1 > z_1$ we have

$$\mathbb{E}[\phi(X_1, s(z_1, z_1; \Phi)) | Y_1 = y_1, Z_1 = z_1] \geq \mathbb{E}[\psi(X_1, s(z_1, z_1; \Psi)) | Y_1 = y_1, Z_1 = z_1]. \quad (5)$$

We depict this intuition in Figure 3 for the case in which Φ represents equity shares and Ψ represents cash payments. The lines represent the equilibrium bids in these two families for a given value of z_1 . On the left is the density $f_{X_1|Y_1=z_1, Z_1=z_1}(x_1)$ of X_1 conditioned on $Y_1 = Z_1 = z_1$. Relative to this density, the expected value of the payments $\phi(X_1, s(z_1, z_1; \Phi))$ and $\psi(X_1, s(z_1, z_1; \Psi))$ are equal. On the right is the density $f_{X_1|Y_1=y_1, Z_1=z_1}(x_1)$ of X_1 conditioned on $Y_1 = y_1$ and $Z_1 = z_1$ for $y_1 > z_1$. The expected value of $\phi(X_1, s(z_1, z_1; \Phi))$ exceeds the expected value of $\psi(X_1, s(z_1, z_1; \Psi))$ relative to this second density, reflecting both the shift of the density given the observation of the larger signal $Y_1 = y_1 > z_1$ and the relative strong steepness of the two families of securities.

5 Conclusions

DeMarzo et al. (2005) identifies the relative steepness of two families of securities as the critical factor in determining which of the two families generates the highest expected revenue for the seller

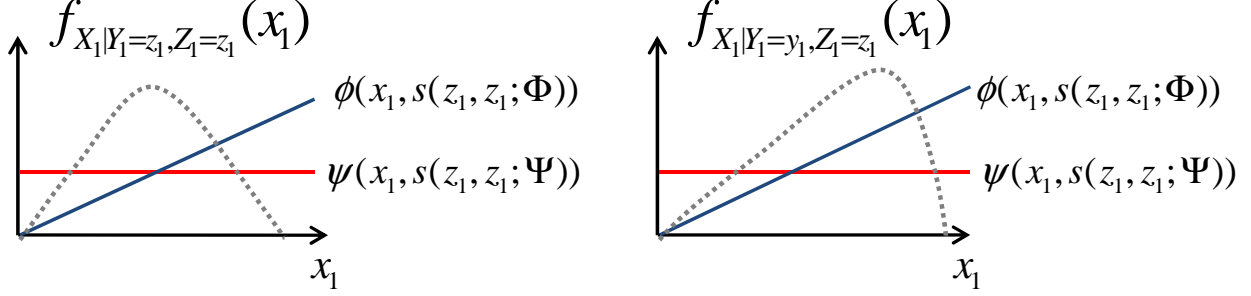


Figure 3: Payments from security bids as functions of X_1 , the pdf of X_1 conditioned on $Y_1 = Z_1 = z_1$ (left), and the pdf of X_1 conditioned on $Y_1 = y_1$ and $Z_1 = z_1$ for $y_1 > z_1$ (right). The family Φ represents equity shares and the family Ψ represents cash payments.

in the second price and the first price auctions. For the second price auction, we first generalize this ranking to include the case of risk averse buyers. We then demonstrate the dependence of this ranking on the underlying positive dependence assumption among values and signals. An example is provided in which positive dependence is relaxed from MLR to FOSD. The pairwise revenue ranking of the common families of securities – debt, equity, and call options – is completely reversed in this example from the ranking of DeMarzo et al. (2005). The cause of this reversal is that positive dependence in the MLR sense globally restricts dependence while positive dependence in the FOSD sense may only restrict it locally; while the local condition of relative steepness is sufficient to rank families in the case of MLR, it must be strengthened in order to obtain a ranking under the less restrictive condition of FOSD. We achieve this by defining a notion of strong steepness and obtaining a ranking of families of securities based on this criterion. This result is significant because FOSD is the property that is most commonly cited in auction theory to motivate an assumption of positive dependence among values and signals.

A Proof of Proposition 1

We start with the following definition:

Definition 7 (Single crossing). A function $g_1(w)$ *single crosses* a function $g_2(w)$ from below if there exists w_0 such that $g_1(w) \leq g_2(w)$ for $w \leq w_0$ and $g_1(w) \geq g_2(w)$ for $w \geq w_0$.

Lemma 5 and Lemma 6 below provide the key steps in establishing Proposition 1.

Lemma 5. Let W be a random variable taking values in some interval J_1 , and let $g_i : J_1 \mapsto J_2, i = 1, 2$ be nondecreasing functions with values in some interval J_2 . Let g_1 single cross g_2 from below and w_0 be the crossing point. Let h be a concave function. Then the following holds:

1. If $\mathbb{E}[g_1(W)] = \mathbb{E}[g_2(W)]$, then $\mathbb{E}[h(g_1(W))] \leq \mathbb{E}[h(g_2(W))]$.
2. If $\mathbb{E}[h(g_1(W))] = \mathbb{E}[h(g_2(W))]$ and $h'(g_1(w_0)) > 0$, then $\mathbb{E}[g_1(W)] \geq \mathbb{E}[g_2(W)]$.

Proof. The first claim is from Ohlin (1969). We therefore turn to the second claim, the proof of which closely follows the proof of the first claim.

Define $F_i(t) \triangleq \mathbb{P}[g_i(W) \leq t]$, $i = 1, 2$, and let $t_0 = g_1(w_0)$. Clearly, F_1 and F_2 are probability measures. If $t < t_0$, the event $g_2(W) \leq t$ implies the event $g_1(W) \leq t$, hence $F_1(t) \geq F_2(t)$. Similarly, if $t > t_0$, the event $g_1(W) \leq t$ implies the event $g_2(W) \leq t$, hence $F_1(t) \leq F_2(t)$.

Since h is concave, it is differentiable almost everywhere (in particular, the right and the left derivatives exist everywhere). Hence, $h(t) = h(t_0) + \int_{t_0}^t h'(r)dr$, where h' can be taken as the right derivative of h . For $i = 1, 2$, regard $g_i(W)$ as a random variable with probability measure F_i . The expected value of $h(g_i(W))$ reduces as follows:

$$\begin{aligned} \mathbb{E}[h(g_i(W))] &= \int_{-\infty}^{\infty} h(t) dF_i(t) = h(t_0) + \int_{-\infty}^{\infty} \int_{t_0}^t h'(r) dr dF_i(t), \\ &= h(t_0) - \int_{-\infty}^{t_0} \int_t^{t_0} h'(r) dr dF_i(t) + \int_{t_0}^{\infty} \int_{t_0}^t h'(r) dr dF_i(t), \\ &= h(t_0) - \int_{-\infty}^{t_0} \int_{-\infty}^r dF_i(t) h'(r) dr + \int_{t_0}^{\infty} \int_r^{\infty} dF_i(t) h'(r) dr, \\ &= h(t_0) - \int_{-\infty}^{t_0} F_i(r) h'(r) dr + \int_{t_0}^{\infty} (1 - F_i(r)) h'(r) dr, \\ &= h(t_0) - \int_{-\infty}^{\infty} F_i(r) h'(r) dr + \int_{t_0}^{\infty} h'(r) dr. \end{aligned}$$

Hence,

$$\mathbb{E}[h(g_1(W))] - \mathbb{E}[h(g_2(W))] = \int_{-\infty}^{\infty} (F_2(t) - F_1(t)) h'(t) dt. \quad (6)$$

Substituting $h(t) = t$ in (6) implies

$$\mathbb{E}[g_1(W)] - \mathbb{E}[g_2(W)] = \int_{-\infty}^{\infty} (F_2(t) - F_1(t)) dt. \quad (7)$$

Since $h'(t)$ is nonincreasing in t , $F_2(t) - F_1(t) \leq 0$ for $t \leq t_0$, and $F_2(t) - F_1(t) \geq 0$ for $t \geq t_0$, we have,

$$(F_2(t) - F_1(t)) h'(t) \leq (F_2(t) - F_1(t)) h'(t_0), \quad \text{for all } t. \quad (8)$$

Combining (6)-(8) results in

$$\mathbb{E}[h(g_1(W))] - \mathbb{E}[h(g_2(W))] \leq h'(t_0) (\mathbb{E}[g_1(W)] - \mathbb{E}[g_2(W)]). \quad (9)$$

The result then immediately follows from (9). \square

Lemma 6. *Let a function g single cross zero from below. Suppose $\mathbb{E}[g(X_1)|Y_1 = \hat{y}_1, Z_1 = z_1] \geq 0$ for some \hat{y}_1 and z_1 . Assuming PD-MLR, $\mathbb{E}[g(X_1)|Y_1 = y_1, Z_1 = z_1] \geq 0$ for any $y_1 > \hat{y}_1$.*

Proof. Since conditioning on $Z_1 = z_1$ plays no role in the above claim, for notational convenience define $h_{X_1|Y_1=y_1}(x_1) \triangleq f_{X_1|Y_1=y_1, Z_1=z_1}(x_1)$, which omits $Z_1 = z_1$. Let w_0 be the point at which g crosses zero from below. Since $h_{X_1|Y_1=y_1}(x_1)/h_{X_1|Y_1=\hat{y}_1}(x_1)$ is increasing in x_1 , $g(x_1) \leq 0$ for $x_1 \leq w_0$, and $g(x_1) \geq 0$ for $x_1 \geq w_0$, we get

$$g(x_1) \frac{h_{X_1|Y_1=y_1}(x_1)}{h_{X_1|Y_1=\hat{y}_1}(x_1)} \geq g(x_1) \frac{h_{X_1|Y_1=y_1}(w_0)}{h_{X_1|Y_1=\hat{y}_1}(w_0)} \quad \text{for all } x_1. \quad (10)$$

Then,

$$\begin{aligned}
\mathbb{E}[g(X_1)|Y_1 = y_1, Z_1 = z_1] &= \int_{\underline{x}}^{\bar{x}} g(w) h_{X_1|Y_1=y_1}(w) dw, \\
&= \int_{\underline{x}}^{\bar{x}} g(w) \frac{h_{X_1|Y_1=y_1}(w)}{h_{X_1|Y_1=\hat{y}_1}(w)} h_{X_1|Y_1=\hat{y}_1}(w) dw, \\
&\geq \int_{\underline{x}}^{\bar{x}} g(w) \frac{h_{X_1|Y_1=y_1}(w_0)}{h_{X_1|Y_1=\hat{y}_1}(w_0)} h_{X_1|Y_1=\hat{y}_1}(w) dw, \\
&= \frac{h_{X_1|Y_1=y_1}(w_0)}{h_{X_1|Y_1=\hat{y}_1}(w_0)} \mathbb{E}[g(X_1)|Y_1 = \hat{y}_1, Z_1 = z_1] \geq 0,
\end{aligned}$$

where the first inequality is from (10). This completes the proof. \square

We can now prove Proposition 1. The seller's revenue if bids are restricted to the family Φ is $\mathbb{E}[\phi(X_1, s(Z_1, Z_1; \Phi))|Y_1 > Z_1]$ and is $\mathbb{E}[\psi(X_1, s(Z_1, Z_1; \Psi))|Y_1 > Z_1]$ if bids are restricted to the family Ψ . To prove Proposition 1, it suffices to show that for any $y_1 > z_1$,

$$\mathbb{E}[\phi(X_1, s(Z_1, Z_1; \Phi))|Y_1 = y_1, Z_1 = z_1] \geq \mathbb{E}[\psi(X_1, s(Z_1, Z_1; \Psi))|Y_1 = y_1, Z_1 = z_1]. \quad (11)$$

From (1), for any z_1 ,

$$\begin{aligned}
\mathbb{E}[u(X_1 - I - \phi(X_1, s(Z_1, Z_1; \Phi)))|Y_1 = z_1, Z_1 = z_1] &= 0 \\
&= \mathbb{E}[u(X_1 - I - \psi(X_1, s(Z_1, Z_1; \Psi)))|Y_1 = z_1, Z_1 = z_1]. \quad (12)
\end{aligned}$$

If either $\phi(x_1, s(z_1, z_1; \Phi)) < \psi(x_1, s(z_1, z_1; \Psi))$ or $\phi(x_1, s(z_1, z_1; \Phi)) > \psi(x_1, s(z_1, z_1; \Psi))$ for all x_1 then (12) would not be true. Hence, $\phi(x_1, s(z_1, z_1; \Phi))$ and $\psi(x_1, s(z_1, z_1; \Psi))$ must cross each other as functions of x_1 . Since Φ is steeper than Ψ , $\phi(x_1, s(z_1, z_1; \Phi)) - \psi(x_1, s(z_1, z_1; \Psi))$ single crosses zero from below. This implies that $x_1 - \psi(x_1, s(z_1, z_1; \Psi))$ single crosses $x_1 - \phi(x_1, s(z_1, z_1; \Phi))$ from below. This, along with (12), and u being concave and increasing, allow for an application of the second part of Lemma 5, and results in the following inequality:

$$\begin{aligned}
\mathbb{E}[X_1 - I - \phi(X_1, s(Z_1, Z_1; \Phi))|Y_1 = z_1, Z_1 = z_1] \\
\leq \mathbb{E}[X_1 - I - \psi(X_1, s(Z_1, Z_1; \Psi))|Y_1 = z_1, Z_1 = z_1]. \quad (13)
\end{aligned}$$

Hence,

$$\mathbb{E}[\phi(X_1, s(Z_1, Z_1; \Phi)) - \psi(X_1, s(Z_1, Z_1; \Psi))|Y_1 = z_1, Z_1 = z_1] \geq 0. \quad (14)$$

Since $\phi(x_1, s(z_1, z_1; \Phi)) - \psi(x_1, s(z_1, z_1; \Psi))$ single crosses zero from below, (14) and Lemma 6 imply

$$\mathbb{E}[\phi(X_1, s(Z_1, Z_1; \Phi)) - \psi(X_1, s(Z_1, Z_1; \Psi))|Y_1 = y_1, Z_1 = z_1] \geq 0, \quad (15)$$

for $y_1 > z_1$. This establishes (11) and the proof is complete. \square

B Proof of Proposition 2

We first explain how the example was devised. Recall the statement of Lemma 6 from the preceding section. The proof of Proposition 1 is an application of Lemma 6 in which for any value of z_1 : (i) $g(x_1) = \phi(x_1, s(z_1, z_1; \Phi)) - \psi(x_1, s(z_1, z_1; \Psi))$, where Φ is a steeper family than Ψ ; (ii) \hat{y}_1 equals z_1 ; (iii) Lemma 6 is applied to derive inequality (15), which is the conclusion that the expected payment by buyer 1 in the event that he trades is greater with the family of securities Φ than with Ψ . Example 1 is constructed with the goal of making this last step false so that the steeper family of securities produces a lower expected payment by buyer 1. The key observation is that while the function $g(x_1)$ is assumed by Lemma 6 to single cross zero from below, this does not preclude $g(x_1)$ from decreasing for values of x_1 below the point at which it crosses zero. In Example 1, a larger value of Y_1 changes the probability density of X_1 only in the interval $[0, 1/3]$, making the values in $[0, 1/3]$ closer to $1/3$ more likely and the values near 0 less likely, while the probability density over $[1/3, 1]$ remains unaffected. If $g(x_1)$ is decreasing over $[0, 1/3]$, then conditioning on a larger value of Y_1 can decrease the expected value of $g(X_1)$ over $[0, 1]$ and thereby reverse the conclusion of Proposition 1. As we show below, this in fact occurs for a range of values of the investment I and for each realization of the signal vector (Y_1, Y_2) in the case in which Φ is the equity family and Ψ is the debt family.

We begin by choosing the investment parameter I to insure that the relevant $g(x_1)$ in the case of debt and equity crosses zero at a value larger than $1/3$. In the case of debt securities, the optimal bid b of buyer 1 when his signal equals zero is determined by the equation

$$\begin{aligned} \mathbb{E}[X_1 - I - \min(X_1, b) | Y_1 = 0] &= 0, \\ \Leftrightarrow \mathbb{E}[X_1 - I | Y_1 = 0] &= \mathbb{E}[\min(X_1, b) | Y_1 = 0]. \end{aligned} \quad (16)$$

With foresight to the use of $g(x_1)$ below, we wish to insure that the optimal bid of buyer 1 when his signal equals zero exceed $1/3$. The left side of (16) is decreasing in I and the right side is nondecreasing in b . At $I = 0.2$ and $b = 1/3$, the left side strictly exceeds the right side. As a consequence, we conclude that there is a value $\bar{I} > 0.2$ such that for all $I < \bar{I}$, the value of b that solves (16) strictly exceeds $1/3$. We therefore fix the investment at some value $\tilde{I} \in (0, \bar{I})$.

Consider an arbitrary realization $(\tilde{y}_1, \tilde{y}_2)$ of the signal vector such that buyer 1 wins, i.e., $\tilde{y}_1 > \tilde{y}_2$. Given \tilde{y}_2 , let b^d denote the bid of buyer 2 when he bids with debt securities and b^e his bid when he bids with equity securities. It is sufficient to prove that

$$\mathbb{E}[b^e X_1 | Y_1 = \tilde{y}_1] < \mathbb{E}[\min(X_1, b^d) | Y_1 = \tilde{y}_1], \quad (17)$$

where the left hand side denotes the seller's expected revenue given $(\tilde{y}_1, \tilde{y}_2)$ in the case of equity securities and the right hand side denotes his expected revenue in the case of debt securities. We are using here the fact that X_1 is independent of Y_2 in this example.

Lemma 4 states that the bids b^d and b^e satisfy:

$$\mathbb{E}[X_1 - \tilde{I} - b^e X_1 | Y_1 = \tilde{y}_2] = 0 = \mathbb{E}[X_1 - \tilde{I} - \min(X_1, b^d) | Y_1 = \tilde{y}_2], \quad (18)$$

$$\Rightarrow \mathbb{E}[b^e X_1 - \min(X_1, b^d) | Y_1 = \tilde{y}_2] = 0. \quad (19)$$

We next apply (18) to bound the bids b^d and b^e . Since $\tilde{I} > 0$, it is straightforward to see that $b^e < 1$. Our foresight in choosing \tilde{I} is now useful: because $\mathbb{E}[X_1 - \tilde{I} - \min(X_1, b^d) | Y_1 = y_1]$ is increasing in y_1 , the solution b^d to (18) is at least as large as its value at $y_1 = 0$ and so $b^d > 1/3$.

Define $g(x_1) \triangleq b^e x_1 - \min(x_1, b^d)$. The function g is decreasing in the interval $[0, b^d]$ and thus decreasing in $[0, 1/3]$. From (19), $\mathbb{E}[g(X_1)|Y_1 = \tilde{y}_2] = 0$. Hence,

$$\begin{aligned}\mathbb{E}[g(X_1)|Y_1 = \tilde{y}_1] &= \mathbb{E}[g(X_1)|Y_1 = \tilde{y}_1] - \mathbb{E}[g(X_1)|Y_1 = \tilde{y}_2] \\ &= \int_0^1 g(w) f_{X_1|Y_1=\tilde{y}_1}(w) dw - \int_0^1 g(w) f_{X_1|Y_1=\tilde{y}_2}(w) dw.\end{aligned}\quad (20)$$

From (2), for $w \in (1/3, 1]$, $f_{X_1|Y_1=\tilde{y}_1}(w) = f_{X_1|Y_1=\tilde{y}_2}(w) = 1$. For $w \in [0, 1/3]$, $f_{X_1|Y_1=y_1}(w) = 1 - y_1 + 6wy_1$ and $g(w) = (b^e - 1)w$. Equation (20) therefore simplifies to:

$$\begin{aligned}\mathbb{E}[g(X_1)|Y_1 = \tilde{y}_1] &= \int_0^{1/3} g(w) (f_{X_1|Y_1=\tilde{y}_1}(w) - f_{X_1|Y_1=\tilde{y}_2}(w)) dw, \\ &= \int_0^{1/3} (b^e - 1)w ((1 - \tilde{y}_1 + 6w\tilde{y}_1) - (1 - \tilde{y}_2 + 6w\tilde{y}_2)) dw, \\ &= \int_0^{1/3} (b^e - 1)(\tilde{y}_1 - \tilde{y}_2)w(6w - 1)dw, \\ &= (b^e - 1)(\tilde{y}_1 - \tilde{y}_2)\frac{1}{54} < 0,\end{aligned}\quad (21)$$

where the last inequality is because $\tilde{y}_1 > \tilde{y}_2$ and $b^e < 1$. This establishes (17) and the proof is complete. \square

C Proof of Proposition 3

The proof is almost the same as the proof of Proposition 1. The main difference is in how the concluding inequality that ranks the expected payments of the winning buyer under different families of securities is derived using PD-FOSD and strong steepness instead of PD-MLR and steepness.

As in the proof of Proposition 1, it suffices to show that (11) holds for any $y_1 > z_1$. The argument in the proof of Proposition 1 implies that $\phi(x_1, s(z_1, z_1; \Phi))$ must cross $\psi(x_1, s(z_1, z_1; \Psi))$ as functions of x_1 . Strong steepness requires that $\phi(x_1, s(z_1, z_1; \Phi)) - \psi(x_1, s(z_1, z_1; \Psi))$ is nondecreasing in x_1 and hence $x_1 - \psi(x_1, s(z_1, z_1; \Psi))$ single crosses $x_1 - \phi(x_1, s(z_1, z_1; \Phi))$ from below. Inequality (14) then follows by the same argument as before, implying:

$$\begin{aligned}\mathbb{E}[\phi(X_1, s(Z_1, Z_1; \Phi)) - \psi(X_1, s(Z_1, Z_1; \Psi))|Y_1 = z_1, Z_1 = z_1] &\geq 0, \\ \Rightarrow \mathbb{E}[\phi(X_1, s(Z_1, Z_1; \Phi)) - \psi(X_1, s(Z_1, Z_1; \Psi))|Y_1 = y_1, Z_1 = z_1] &\geq 0.\end{aligned}$$

The last inequality which proves the result is from an application of Lemma 2, using the fact that $\phi(x_1, s(z_1, z_1; \Phi)) - \psi(x_1, s(z_1, z_1; \Psi))$ is nondecreasing in x_1 (i.e., strong steepness) together with PD-FOSD. \square

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